MAT 402 S26
PROBLEM SET 2

I. Consider the system with one degree of freedom defined by a twice-differentiable potential function $U : \mathbb{R} \to \mathbb{R}$. That is to say, the position $x(t)$ of this mechanical system satisfies the differential equation

$$\ddot{x}(t) = -U'(x(t)).$$

Suppose that at time t_o the position and velocity of the mechanical system are $x(t_o) = x_o$ and $\dot{x}(t_o) = v_o$ repsectively, and at some later time t_1 the position and velocity of the mechanical system are x_1 and v_1 respectively.

a. Prove that $\frac{v_1 \cdot v_1}{2} + U(x_1) = \frac{v_o \cdot v_o}{2} + U(x_o)$.

b. Prove that if $x(t) \neq x(s)$ for all distinct $s, t \in [t_o, t_1)$ then

$$t_1 - t_o = \int_{x_o}^{x_1} \frac{dy}{\sqrt{2(E - U(y))}},$$

where $E = \frac{v_o \cdot v_o}{2} + U(x_o)$.

II. Consider a system with one degree of freedom defined by a twice-differentiable potential function $U : \mathbb{R} \to [-1, \infty)$. Show that the solutions of this mechanical system exist for all real time, no matter what the initial condition.

III. Let E be a Euclidean vector space of dimension 3 and let $\langle v, w\rangle$ denote the inner product of $v, w \in E$. Fix an orientation $\mathfrak{f}$ for E.[1] Recall that the vector product

$$E \times E \ni (v, w) \mapsto [v, w] \in E$$

is defined by the following three properties.

i. $\langle v, [v, w]\rangle = \langle w, [v, w]\rangle = 0$

ii. $|[v, w]| = \text{Area}\left(\{sv + tw \in E\ ;\ (s, t) \in [0, 1] \times [0, 1]\}\right)$

iii. If $[x, y] \neq 0$ then the ordered triple $(x, y, [x, y]) \in \mathfrak{f}$.

Prove that for all $u, v, w \in E$

(a) $[v, w] = -[w, v]$

(b) $[u, av + bw] = a[u, v] + b[u, w]$.

(c) $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$.

(A vector space $\mathfrak{a}$ with an operation $\mathfrak{a} \times \mathfrak{a} \ni (x, y) \mapsto [x, y]$ satisfying (a), (b) and (c) is called a *Lie algebra*.)

IV. with the notation of Problem III, suppose (e_1, e_2, e_3) is a positively oriented orthonormal basis for E. Find a formula for the vector product

$$[a^1e_1 + a^2e_2 + a^3e_3, b^1e_1 + b^2e_2 + b^3e_3].$$

Justify your answer rigorously.

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[1] Recall that an orientation for E is an equivalence class of ordered bases ented triples (u_1, u_2, u_3) for E, where (u_1, u_2, u_3) and (v_1, v_2, v_3) are equivalent if the 3×3 real matrix A defined by

$$v_i = A_{i1}u_1 + A_{i2}u_2 + A_{i3}u_3, \quad i = 1, 2, 3,$$

has positive determinant.

V. Consider the central force field $F : \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(x) = \frac{2A}{|x|^3}x.$$

Suppose that the particle moving under the influence of this force, i.e., whose position $x(t)$ at time $t \in \mathbb{R}$ satisfies the equation

$$\ddot{x}(t) = F(x(t)),$$

is at the point $(1, 0, 0)$ at time $t = 0$, and has velocity $(0, 1, 1)$ at time $t = 0$.

a. Show that for all $t \in \mathbb{R}$ the position $x(t)$ of the particle lies in 2-dimensional subspace $\Pi \subset \mathbb{R}^3$.

b. Let $r(x)$ and $\theta(x)$ be the polar coordinates, with pole at the origin, of a point x in the plane of motion Π determined in Part a, normalized so that $r(1, 0, 0) = 1$ and $\theta(1, 0, 0) = 0$. Let

$$R(r) := r(x(t)) \quad \text{and} \quad S(t) := \theta(x(t))$$

be the polar components of $x(t)$ at time t. Show that there is a constant $E \in \mathbb{R}$ such that

$$\cos S(t) = \frac{1 - AR(t)}{R(t)\sqrt{E + A^2}} \quad \text{for all } t \in \mathbb{R}.$$

c. Let E be the constant in part b. In terms of the polar coordinates (r, θ) in the plane Π defined in part b, let

$$C(A) := \left\{ x \in \Pi \; ; \; (r(x), \theta(x)) \; ; \; \cos\theta(x) = \frac{1 - Ar(x)}{r(x)\sqrt{E + A^2}} \right\}.$$

For which values of A is the set $C(A) \subset \mathbb{R}^3$ unbounded?